NONLINEAR EIGENVECTOR CENTRALITY

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Eigenvector centrality:

Eigenvector $\boldsymbol{u} \geq 0$ that quantifies the importance of the nodes in a network

$$M_G \boldsymbol{u} = \lambda \boldsymbol{u}$$

M_G graph matrix e.g. Adjacency, Random Walk, PageRank

For the adjacency matrix

$$u_i \propto \sum_j A_{ij} u_j = (A\boldsymbol{u})_i$$

Importance of *i* is **linearly proportional** to the importances of its neighbors

Main drawbacks:

1. Certain **importances are not linear**
2. The solution **may not be unique**

General **nonlinear proportionality relations**:

$$u_i \propto \sum_j A_{ij} f(u_j) \qquad u_i \propto \sum_{jk} A_{ijk} f(u_j, u_k)$$

Nonlinear eigenvector centrality:

$$\mathcal{M}_G(\boldsymbol{u}) = \lambda \boldsymbol{u}$$

THEOREM

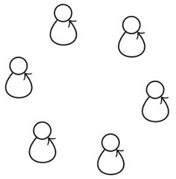
Let $\partial\mathcal{M}_G$ denote the Jacobian of $\mathcal{M}_G$. If

$$|\partial\mathcal{M}_G(\boldsymbol{x})|\boldsymbol{x} < \mathcal{M}_G(\boldsymbol{x}) \qquad (entrywise)$$

for all $\boldsymbol{x} > 0$, then there exists a unique nonlinear eigenvector centrality $\boldsymbol{u} > 0$, $\mathbb{1}^T\boldsymbol{u} = 1$ such that $\mathcal{M}_G(\boldsymbol{u}) = \lambda\boldsymbol{u}$ and we can compute $\boldsymbol{u}$ to an arbitrary precision using a nonlinear Power Method.

$$A\boldsymbol{u} = \boldsymbol{u}, \ \forall \boldsymbol{u}$$

So the centrality is **highly ambiguous**



$$A = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

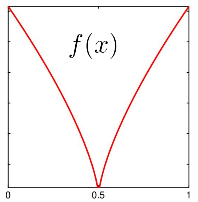
$$\mathcal{M}_G(\boldsymbol{u}) = A\boldsymbol{u}^\alpha = \boldsymbol{u} \iff \boldsymbol{u} = \mathbf{1}$$

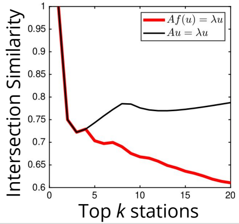
$$|\partial\mathcal{M}_G(\boldsymbol{x})|\boldsymbol{x} = \alpha A \boldsymbol{x}^\alpha$$

For $\alpha \in (0,1)$ only one nonlinear centrality: the "correct one"

Example: Passengers prefer to use a station over another if it is well connected to important stations but it is surrounded by minor stations

$$u_i \propto \textstyle\sum_j A_{ij} f(u_j) \qquad f(x) = |x - \tfrac{1}{2}|^\theta$$

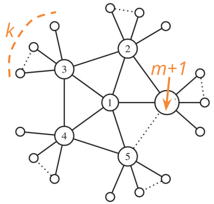


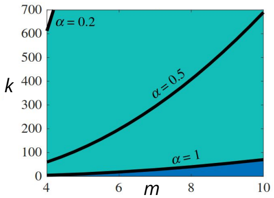


Example: Triangle-aware eigenvector centrality

$$u_i \propto \alpha \textstyle\sum_j A_{ij} u_j + (1-\alpha) \sum_{jk} \mathcal{A}_{ijk} (u_j^p + u_k^p)^{1/p}$$

$\mathcal{A}_{ijk} = 1$ if ijk form a triangle





Values of *m* and *k* for which $u_1 > u_2$

[1] Arrigo, Higham, Tudisco, *Proc. R. Soc. A* ***476***, **2020**; [2] Tudisco, Higham, *SIAM Math Data Science* **1**, **2019**; [3] Gautier, Tudisco, Hein, *SIAM Matrix Anal. Appl* **40**, **2019**;